Self-organized criticality and phase locking in models of production and inventory dynamics

Preliminary version

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This paper proposes an extension of the inventory production model developed in Bak, Chen, Scheinkman, and Woodford [1993]. We show how the Pareto-Levy type of aggregate distributions emerge in a wider class of models and propose an analysis of a class of self-organizing criticalities. It shows that, in these models, self-organized criticalities emerge naturally from the state randomization produced by order propagation. It suggests that a basic randomization process is behind many other self-organized criticalities.

Introduction

The concept of self-organizing criticalities (SOC) was introduced as a physical model to explain a vast class of self-similar behavior often found in physical systems. Bak, Tang, and Wiesenfeld [1987] developed as a prototype model of self-organizing criticalities a sandpile model.

The sandpile model is a cellular automata that mimics the behavior of a real sandpile. If grains of sand are randomly added to a sandpile, the pile self-organizes into one of a specific and stable steepness, regardless of its initial steepness. In addition, the stable shape is maintained in a situation of criticality, producing avalanches of any size. The distribution of avalanche size is of the Pareto-Levy type and shows invariance of scale.

Since the introduction of the first sandpile model, other structures showing similar behavior have been studied. In particular, Carlson, Chayes, Grannan, and Swindle (CCGS) [1992] showed that self-organizing criticalities manifest themselves in diffusion-type processes if the diffusion coefficient manifests singularities. They demonstrate how this type of diffusion arises naturally as a limit behavior of a linear sandpile.

Bak, Chen, Scheinkman, and Woodford (BCSW) [1993] used the Bak sandpile model to build a model of a production and inventory economy that shows large endogenous fluctuations, even in the limit of external stimuli of constant mean. They thus demonstrated that random shocks to the economy do not average out in aggregate but might produce significant aggregate fluctuations.

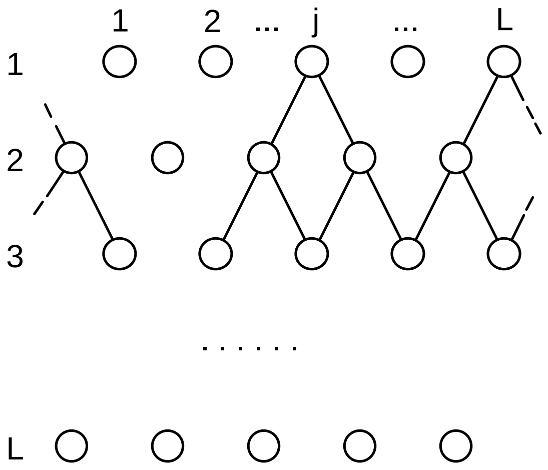

Fig. 1. The BCSW model.

BCSW considered a simple square economy, schematically shown in Fig. 1, made of *L* layers with *L* nodes each. Each node is a producer/warehouse that produces/stores only one type of good. A warehouse can store only one unit. Each producer is characterized by a nonlinear production schedule so that it produces either zero or two units of product. Each producer receives orders from the two nearest producers one layer above and gives orders only to the two nearest producers one layer below. Input orders (final consumers orders) are given to the first layer. In a finite model there is a flow of orders from the L_{th} layer (raw material orders). Moreover, the nodes are arranged on a cylinder to avoid border effects.

BCSW showed that each final consumer order might create production avalanches whose sizes follow a Pareto-Levy type of distribution. Using the special aggregation properties of the Pareto-Levy stable distributions, they then showed that the aggregate behavior follows the same distribution. Even in the limit of an infinite economy, aggregate production - scaled by a suitable factor of scale such that the average of total production remains constant when the economy approaches infinity - shows the same fat-tailed distribution and scale invariance.

Generalizations

Self-organizing criticalities are characterized by several key properties. First, they are phenomena of statistical equilibrium. The equilibrium, however, is critical as there are fluctuations of any size, driven by random shocks, around the equilibrium position. Second, the critical equilibrium is self-organizing as the system automatically reaches the critical equilibrium position from any initial configuration. Third, fluctuations show self-similar behavior with fat-tailed distributions of the Pareto-Levy type and power spectrum decaying as ω^{-b}. This type of behavior has to be distinguished from other types of critical equilibrium that depend on fine tuning some parameters of the system.

The sandpile model as well as the BCSW model are highly idealized SOC models. To generalize and widen the applicability of these types of models, different strategies might be adopted. One strategy is to analyze in detail more complex models, an endeavour that might prove very difficult for realistic models. A second strategy is simply to accept the existence of *1/f* noise as well as spatial self-similar structures. Mathematical analysis of systems can be performed assuming this type of noise and/or spatial fractal structure.

For some broad class of structures, however, it might be possible to analyze how 1/*f* noise is generated from white noise and its propagation. In fact, SOC shows how systems subject to random shocks self-organize into systems that respond to white noise producing 1/*f* noise. We generalize by allowing a broader class of model topology and by adding randomness at each stage.

Our line of reasoning is the following. The formation of avalanches in the BCSW model as well as in our models is due to spatial correlation phenomena. There are avalanches because of long-range correlations between sites. Such long-range correlations are not due to some direct physical or economic influence over long ranges; they are produced because the set of states is repeatedly mapped onto itself by the propagation mechanism. This self-mapping generates a random fractal structure of states. The random fractal structure of states is due to randomization of states at a critical density of inventory occupancy. It is responsible, in turn, for the emergence of power law distributions in response to random shocks.

The mathematics behind long-range memory models

We would like to resume the mathematical ideas that we will use in building our model. The basic insight is that long-memory processes, i.e. processes with long-range correlations, can be generated as moving averages of independent Gaussian processes if averages are taken with appropriate scaling factors. This means that we can translate deterministic self-similar power laws into statistical self-similar distributions starting from Gaussian distributions. Alternatively, *1/f* noise might be generated from white noise (fractional processes might be generated from standard Brownian motions) if there is some power law scaling.

In standard analysis of diffusion processes, one assumes as given a white noise or a standard Brownian motion. White noise is a complex object that needs the theory of distributions to be defined. We only consider Brownian motions.

A classical Brownian motion $B(t)$ is a Gaussian process with independent increments. Its paths are continuous functions of time. In addition, it is a self-similar process; both its path and its graph are fractals of Hausdorf dimension respectively of 2 and $2_{1/2}$. Starting from Brownian motions as the primary source of uncertainty, it is possible to construct generalized stochastic integrals, or Ito integrals, defined as:

$$X(t)=\int_0^{\tau} \mu(s)ds + \int_0^{\tau} \sigma(s)dB(s).$$

An Ito diffusion, or simply a diffusion, is an Ito integral that solves the Stochastic Differential Equation:

$$dX=\mu(t,X)dt+\sigma(t,X)dB(t).$$

Brownian motions and related diffusions can be generalized in two directions. The first maintains path continuity, but gives up independent increments. This results in fractionally integrated processes. The second maintains independent increments, but gives up path continuity by replacing Gaussian distributions with stable distributions. Both generalizations are useful in connection with the phenomena of self-organized criticalities.

Fractionally integrated Brownian motion was introduced by Mandelbrot and Van Ness (1968). Comte and Renault (1993) developed a more general mathematical theory of fractionally integrated or differentiated processes that parallels the theory of Ito diffusions.

A continuous-time, fractional Brownian motion (FBM) of order α, $-1/2< \alpha <1/2$, is a process $B_\alpha(t)$ defined as follows:

$$B_\alpha (t) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\tau (t - s)^\alpha \, dB(s)$$

A FBM is therefore a stochastic integral weighted with the factor $(t\text{-}s)^\alpha$. There are questions as regards the origin of integration as the weighting factor is not defined at zero. A FBM is therefore generally defined by its increments:

$$B_\alpha (t) - B_\alpha (0) = \frac{1}{\Gamma(\alpha + 1)} \int_{-\infty}^\tau (t - s)^\alpha \, dB(s) - \frac{1}{\Gamma(\alpha + 1)} \int_{-\infty}^0 (-s)^\alpha \, dB(s)$$

Comte and Renault showed that the two definitions are asymptotically equivalent. They generalized FBMs by defining continuous-time fractionally integrated processes of order α, $-1/2< \alpha <1/2$, as the following processes $X(t)$:

$$X(t) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\tau (t - s)^\alpha A(t - s) dB(s)$$

They also showed how to give meaning to a stochastic integral of the form:

$$X(t) = \int_0^\tau D(t - s) dB_\alpha (s)$$

which is the equivalent of an ordinary stochastic integral except that it is taken with respect to a fractionally integrated Brownian motion.

In discrete time, fractionally integrated processes become the fractionally integrated sequences that were defined by Granger and Joyeux (1980) and Hosking (1981):

$$X(i) = \sum\nolimits_k \frac{\Gamma(k - \alpha)}{\Gamma(-\alpha)\Gamma(k + 1)} \varepsilon(i - k)$$

where the $\varepsilon(i\text{-}k)$ are IID variables of finite variance. In addition, for large values of k the coefficients are approximately proportional to $k^{-\alpha-1}$.

Mandelbrot and Van Ness (1968), and Comte and Renault (1996) proved that FBMs and fractionally integrated processes are long-memory processes, characterized by power spectra and correlation functions that decay with algebraic laws. In particular, it can be demonstrated that fractionally integrated processes of order α have an autocorrelation function that decays with exponent $2\alpha-1$ and power spectra that decay with exponent -2α.

Fractional Brownian motions are self-similar process in the sense that the probability distributions of $B_\alpha(ut)/u^{\alpha+1/2}$ and $B_\alpha(t)$ are identical. In addition, the graph of a fractionally integrated process of order α has Hausdorf dimension $2-\alpha$.

Parallelling the standard theory of SDEs, Comte and Renault demonstrated how a fractionally integrated process, could be represented as the solution of a fractional stochastic equation, which is the equivalent of a SDE with the usual dB term replaced by a fractionally Brownian motion.

Comte and Renault (1993) also demonstrated that long-memory continuous-time processes could be obtained as the limit sum of standard diffusions if drifts and volatilities are independent and distributed as Gamma functions.

The generation of power laws

Key to our argument is the fact that power scaling appears naturally in random structures. Consider, for instance, a random square array of points. This type of random square array is central in percolation models. As shown in Stauffer [1985], above the percolation threshold density, the array is Euclidean. At the percolation density, however, the array is a fractal and the distribution of length of connected sets of points follows a power law. In addition, below the percolation density, the array is a fractal at scales below the correlation length.

We consider triangular bond percolation lattices. As demonstrated in Stauffer [1985], for these triangular lattices the bond percolation threshold and the exponent of the correlation length are known exactly. They are respectively ½ and 4/3. At this critical density the lattice is a random fractal. The distribution of the length of connected links follows a power law.

A generalized model

Taking the infinite limit for the economy is necessary if the system is to show true self-similar behavior at every scale of length. We maintain this assumption though we later suggest a possible different framework of analysis that does not require an infinite system.

The mathematical analysis performed by BCSW critically depends on the L x L square structure of the model and on the rigid connections between firms. Our previous argument, however, shows that the rigid square structure and topology of the model is unnecessary. We therefore allow for more general links between firms and for some randomness in the way orders are transferred from one layer to the next.

The configuration of our model is the following: the system is made up of a finite number M of layers. Each layer is made of L nodes. M might be larger than L and it will become infinite in the limit. Each node has the same configuration as in the BCSW model. In brief, each node represents a firm that produces and stores one good.

Each firm is characterized by a nonlinear production schedule such that it either produces zero or two units. Its warehouse can store only one unit of the good. If, on receiving an order, the firm has one unit in its warehouse, it satisfies the order from its inventory which is thus emptied. If the firm has no unit in inventory, it produces two units. One of the units is delivered to satisfy the order, the other remains in inventory. When it produces two units, a firm simultaneously issues two orders to two firms randomly chosen in the next layer.

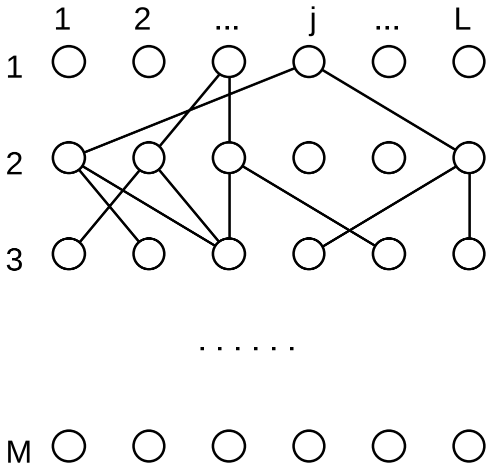

Fig. 2 - The layout of the model

As observed in BCSW, the nonlinearity of the production schedule plays a fundamental role in the model. We could consider more elaborate nonlinear production schedules that would not change the fundamental results while adding complexity to the model.

We depart from BCSW by allowing random connections between each layer and the following. Although we allow each firm to pass orders only to firms in the next layer, we randomly select connections. Figure 2 illustrates the layout of the model.

The zeroth layer receives a constant number of orders per unit time on randomly chosen input nodes. Suppose that the system receives N_0 orders per unit time Δt randomly allocated to each node. Each set of N_0 orders propagates through subsequent layers, generates N_i orders per layer i and changes stock occupancy.

Analysis of the model's long-memory effects

We sketch an analysis of the model as follows. Order propagation changes the site occupancy. Suppose that L is finite and that a proportion p of firms have an unit in inventory. The probability that an order hits a site with a unit of product in inventory is p. The site is then emptied. As a consequence, the long range occupancy ratio of sites is ½.

Therefore, the probability that an order from layer i to layer i+1 hits an occupied site is ½. The path of a single order through the system is therefore isomorphic to a percolation path at the critical bond threshold ½. As seen above, the correlation exponent of such a lattice is 4/3. Therefore, the probability that an order reaches the l_{th} layer is $l^{-4/3}$. This is in agreement with the BCSW model.

Let's now derive long-memory properties of the order-flow process. If the random variable $N(i)$ represent orders at the i_{th} layer, we have to show that $N(i)$ is a fractionally integrated process. The probability that orders pass the i_{th} layer scale as $l^{-\alpha}$. As a consequence, the average size of the orders that go through the i_{th} layer scale as l^{α}. Consider now a flow of orders through the spatial parameter i. $N(i)$ can be written as $\Sigma\Delta N(i-k)$. But $\Delta N(i)$ scales as $\alpha l^{\alpha-1}$. If we now consider i as a continuous parameter, we can think of orders as a continuous flow that can be represented by a fractionally integrated process:

$$N(i) = N_0 + Cost. \int_0^{\tau} (t-s)^{\alpha}\, dB(s)$$

This shows that orders, and therefore production, are subject to long range correlation.

Simulation results

Results of simulations show, in fact, that the tails of the distribution of aggregate production follow a power law. We have conducted several simulations with systems of different sizes. The figures 3 and 4 show results for a system with 200 layers of 20 nodes each, stimulated with 10^6 single inputs on layer 1. The production values are divided by 100, while the frequency values show directly the number of stimulations corresponding to the production.

All the simulations performed show a tail behavior that might be reasonably well approximated by a power law. This fact is perhaps more clearly seen in the log-log plot of the same curve, shown in Figure 4. The initial section of the curve, i.e. the distribution of orders that "die" early, is quite far from a power law distribution.

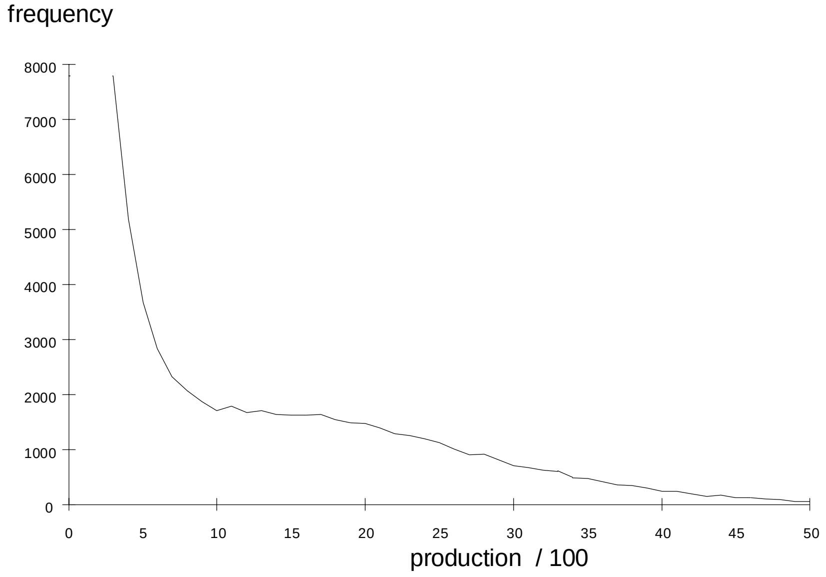

Fig. 3. Frequency distribution of production for a system of 100 x 20 "firms".

Simulations show a characteristic behavior with a short-range peak before the power law behavior prevails. We believe that this behavior might reflect the initial stages of the process which is a standard diffusion process.

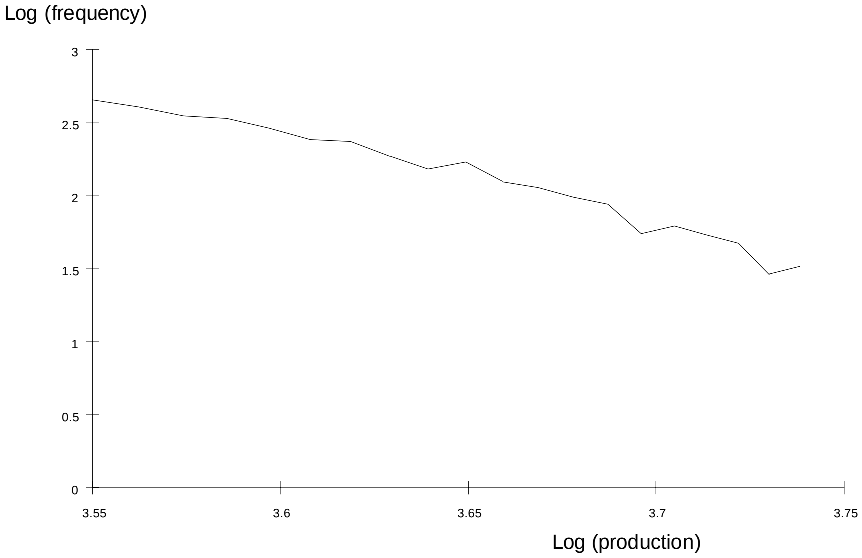

Fig. 4. log-log plot of frequency distribution tail of the data in Fig. 3.

Stable processes and fractal random walks

In the above, we looked at the formation of long-range spatial correlations in a flow of orders subject to a non-linear production/inventory dynamics. These correlations result in a production schedule whose tail distribution follows a power law. We can now look at how power laws might propagate.

Consider, for instance, a system that is characterized by a power law distribution in the conditional probability $P(N_i/N_{i-1})$ distribution of orders. In the previous models, these conditional distributions were Gaussian. This means that orders might experience jumps of any size between layers. To address this problem, the related but conceptually different mathematical tools of fractional calculus have to be used.

Each Brownian motion $B(t)$ is associated to a Cauchy partial differential equation $\vartheta p/\vartheta t=\vartheta^2 p/\vartheta^2 x$. The fundamental solution of such an equation is the probability distribution $p(x,t)$ of $B(t)$ starting at some initial point.

Fractional calculus is based on defining the fractional integral and derivative of deterministic functions through the associated space of Fourier transforms. The connection of fractional calculus with probability theory is the following. Feller has shown that all the stable distributions can be generated as solutions of the equation:

$$dp/dt=D_\theta^\alpha$$

where D_θ^α is fractional differential operator. As shown in Gorenflo and Mainardi [1998], the above equation can be discretized as:

$$p_i(t_{n+1}) - p_i(t_n) = \text{Cost. } \Sigma_k p_k p_{i-k}(t_n)$$

where the k-step transition probabilities p_k are proportional to $k^{-\alpha-1}$.

Let's look at a generalized system that is not necessarily a square system. We can associate to each node a probability $P(i,N)$ which is the probability that layer i issues N orders. For each layer, the probability distribution $P(N_i=n)$ is given by:

$$P(N_i=n)= \Sigma_m P(N_i=n/N_{i-1}=m)\ P(N_{i-1}=m).$$

In fact, the probability that a layer i produces n orders is given by the probability that it receives m orders multiplied by the corresponding transfer probabilities and summed over all possible values of m. Assuming that we know how to factorize the transition probabilities, the above expression gives the link required to understand the evolution of probabilities in space.

The type of distributions that arise depends critically on the coefficients of the above sum, i.e., on the conditional probabilities of jumping from different levels of orders. If we recall the above discretization, we see that if coefficients scale as $k^{-\alpha-1}$ then the above equation can be considered a numerical approximation to the Feller space-fractional diffusion equation $dp/dt=D_\theta^\alpha$ that generates all the Pareto-Levy stable distributions. Starting from any initial distribution, including a constant, a system characterized by the above conditional probability distributions evolves as a sequence of Pareto-Levy stable distributions in the continuous parameter i.

Phase locking and stochastic resonance

All the above analyses critically depend on the size of systems. The emergence of scaling laws is due to the fact that the repeated application of the order propagation dynamics produces a randomization of states at a critical density. If systems are of a finite size, scaling is by necessity truncated.

Following an idea of the economist Krugman, we want to explore a different set of concepts to explain dependencies between economies or sectors. Krugman [1996] observes that the size of exchanges between large economic blocks, for instance Europe and the USA, is of small size in comparison with the size of their internal economies. Still, these exchanges might have large effects on the respective internal economies. Krugman suggests that there might be a synchronization effect, similar to the phase locking effect between physical resonant systems. Using our previous models, we have performed a number of simulations with weak interaction between economies.

We simulated two systems having the same number of “firms” (200 x 20), both stimulated randomly 8092 times. For each system, we recorded the sequence of activities (total system production) in response to stimuli. Let $\boldsymbol{P}^1$ and $\boldsymbol{P}^2$ denote these sequences. Taking advantage of the free connection structure of our models, we let a small fraction of orders at each layer be randomly exchanged between the two economies. We then computed the correlation between the activity sequences of the two economies, varying the coupling. Such correlations are the dot product between $\boldsymbol{P}^1$ and $\boldsymbol{P}^2$.

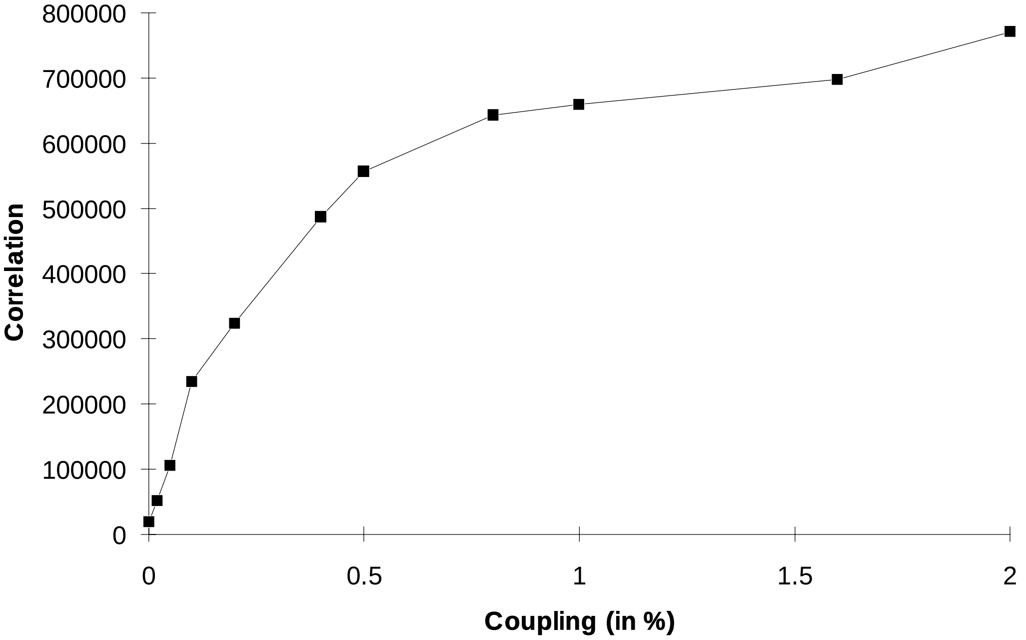

Fig. 5. Correlation between two systems of 200 x 20 “firms”, as a function of their coupling.

The results of the simulations are reported in Fig. 5. It is worth noting that for these systems with these stimuli, the correlation values between non correlated systems are of the order of 10,000-30,000, while autocorrelations are of the order of 1,000,000. As seen in the figure, even a very small coupling coefficient, given by the percentage of orders exchanged, results in significant correlations between the two processes.

We are presently working at building a theoretical explanation of correlations through the concepts of stochastic resonance that will be presented in a forthcoming paper. Stochastic resonance is a set of mathematical tools that permit to study the behavior of multistable systems in the presence of noise when small periodic perturbations are added.

It was shown in different contexts that driving a noisy system with small periodic perturbations might produce effects that couple in nonlinear ways with noise-induced effects. For instance, the rate of transitions in a bistable system subject to white noise shows a typical resonance spectrum. This type of phenomena has been investigated theoretically and experimentally in a large number of physical systems.

Our present work is aimed at extending this framework to randomly fluctuating coupled systems. In these systems there is no periodic driving force. However, random fluctuations might still be synchronized under appropriate conditions. Suppose, for instance, that two economies subject to random aggregate fluctuations are weakly coupled by economic exchanges. The two economies might then experience a significant synchronization of economic cycles.

Conclusions

In this work, we have generalized the BCSW model. We have shown that economic variables such as orders and production might propagate through the economy with significant long-range correlations due to the distribution of the state of the economy which is driven by the propagation itself.

We believe that the contribution of this work is to show that, for a broad class of models, self-organized criticalities originate from a simple mechanism of randomization at a critical threshold. The system shows SOCs simply because order propagation generates a random fractal of states at a critical density. The density itself is a consequence of the specific production schedule chosen. We suspect that a simple randomization mechanism is behind broader classes of SOCs.

We have also indicated that other types of long-range correlations might be at work, in particular synchronization phenomena that might be explained within the framework of stochastic resonance. Stochastic resonance might prove to be a fundamental paradigm to explain synchronization of economic variables across the economy.

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